

A Graphical Construction of the $sl(3)$ Invariant for Virtual Knots

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Abstract

We construct a graph-valued analogue of the Homflypt $sl(3)$ invariant for virtual knots. The restriction of this invariant for classical knots coincides with the usual Homflypt $sl(3)$ invariant, and for virtual knots and graphs it provides new information that allows one to prove minimality theorems and to construct new invariants for free knots. We formulate this new invariant for virtual braids and show that this leads to the construction of a trace function on the virtual Hecke algebra. Finally, we show that the Penrose coloring bracket is a special case of the Kuperberg bracket and we raise new questions about the extension of the present work.

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1 Introduction

This paper studies a generalization to virtual knot theory of the Kuperberg $sl(3)$ bracket invariant. Kuperberg discovered a bracket state sum for the $sl(3)$ specialization of the Homflypt polynomial that depends upon a reductive graphical procedure similar to the Kauffman bracket but more complex. We note that the Kuperberg bracket can be uniquely defined and generalized to virtual knot theory via its reductive graphical equations. But these equations reduce to scalars only for the planar graphs from classical knots. In the case of virtual knots, there are unique graphical reductions to linear combinations of reduced graphs with Laurent polynomial coefficients. Because there are unique graphical reductions, these graphs feature strongly in the invariants. Let us call these graph polynomials. The ideal case, sometimes realized, is when the topological object is itself the invariant, due to irreducibility. When this happens one can point to combinatorial features of a topological object that must occur in all of its representatives. Thus we construct an invariant of virtual knots that takes values in graph-polynomials. This extended Kuperberg bracket specializes to an invariant of free knots and allows us to prove that many free knots are non-trivial without using the parity restrictions we had been tied to before. Furthermore, the extended Kuperberg bracket has an interpretation as a trace on a virtual Hecke algebra and so we have another aspect to this investigation.

This article arose through our discussions of new possibilities in virtual knot theory and in relation to advances of Manturov using parity in virtual knot theory, particularly in the area of free knots. Manturov was the first person to show that many free knots are non-trivial by using the parity bracket (described below). Free knots are the most rarefied of the knot theory variants that we consider. A free knot is a Gauss diagram with only chords and without signs on the chords or orientations on them. Such Gauss diagrams are taken up to Reidemeister moves and they underlie the structures of virtual knot theory. A standard virtual knot maps to a free knot by forgetting much of its structure, but if the underlying free knot is non-trivial then the overlying virtual knot is also non-trivial. Thus free knots are fundamental to the understanding of virtual knots.

The aim of the present article is to extend the Kuperberg combinatorial construction of the quantum $sl(3)$ invariant for the case of virtual knots. The constructed invariant has the following properties.

1. It coincides with the usual $sl(3)$ quantum invariant in the case of the classical knots.
2. It does not change under virtualization (an operation on virtual knots and their projections – defined in the next section) ; its specification at $A = 1$ gives rise to an invariant of free knots.
3. For virtual knots, that are complicated enough, the new invariant is valued in a certain module whose generators are graphs.
4. The invariant produces many new examples of minimality in a strong sense for free knots in the flavour of the parity bracket, [6]. But it goes beyond simple parity and can discriminate certain free knots that have only even crossings.

In the paper [4], there is a model for the $sl(n)$ -version of the Homflypt polynomial for classical knots. This model is based on patterns of smoothings as shown in Figure 1.

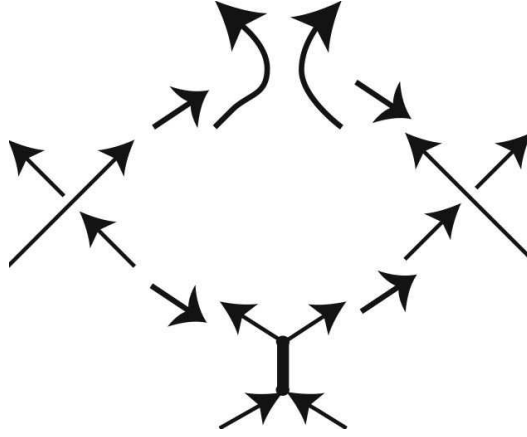


Figure 1: **Murakami-Ohtsuki-Yamada relation for $sl(n)$**

These patterns suggested to us the techniques we use in this paper with the Kuperberg bracket, and we expect to generalize them further. In this method, the value of the polynomial for a knot is equal to the linear combination of the values for two graphs obtained from the knot by resolving the two crossings as shown above. We do not indicate concrete coefficients, because we are not going to use them in the rest of this paper. Below, we adopt the usual convention that whenever we give a picture of a relation, we draw only the changing part of it; outside the figure drawn, the diagrams are identical.

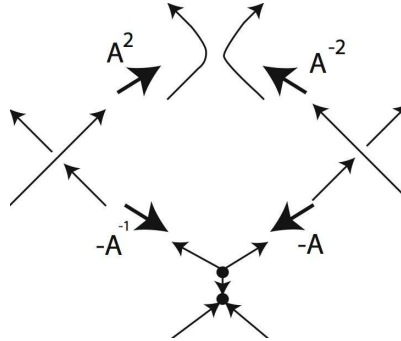


Figure 2: **Kuperberg's relation for $sl(3)$**

For the case of the $sl(3)$ knot invariant, one can use another relation, shown in Figure 2 instead, see [3]. This means that the left (resp., right) picture of (2) is resolved to a combination of the upper and lower pictures with coefficients indicated on the arrows. Note that in the case of Figure 2 we do not discriminate any specific edges of the resulting graph. The advantage of Kuperberg's approach is that graphs of this sort which can be drawn on the plane can be easily simplified, by using further linear relations, to collections of Jordan curves, which in turn, evaluate to elements from $\mathbb{Z}[A, A^{-1}]$. For planar graphs, these reductions continue all the way to scalars. In the case of non-planar graphs, there is no immediate way to resolve such graphs to linear

combinations of collections of circles. We take it as an extra advantage of this approach that the non-planar resolutions leave irreducible graphs whose properties reflect the topology of virtual knots and links.

In a sequence of papers [1, 2] M.Khovanov and L.Rozansky constructed a way of categorifying the general $sl(n)$ Homflypt-polynomial: they constructed a complex whose Euler characteristic is equal to the Homflypt polynomial (in some other normalization). A careful look at this paper shows that their construction can be used for virtual knots as well; so, it gives rise not only to the Homflypt polynomial for virtual knots, but also to its categorification. However, their construction is far from being combinatorial: to construct a polynomial, one needs to calculate dimensions of huge tensor products modulo complicated relations. It can be understood that the constructions [1, 2] are invariant under the virtualization move (see definition ahead).

In the present paper we enhance the Homflypt $sl(3)$ invariant by using the following observation: *if a trivalent graph is complicated enough so that it admits no further simplification, it can be evaluated as itself*. Then the obtained “polynomial” $sl(3)$ invariant will be valued not just in Laurent polynomials in one variable A , but in a larger ring where trivalent graphs act as variables. A key point here is that it can be the case that a topological object such as a free knot is its *own* invariant! This is what happens when we meet irreducibility in the graphical expansion of our invariant. Then it is possible for the expansion to simply stop at the object itself. We illustrate this phenomenon for the parity bracket in Section 3. The same rigidity occurs when the non-planar graphs in the expansion of the generalized Kuperberg bracket are irreducible. Then these graphs are valued as themselves, rather than as polynomials in other graphs.

There are evident advantages to this approach:

1. For all graphs and knots that satisfy all the $sl(3)$ skein relations listed above, and the virtualization relations, the Homflypt $sl(3)$ invariant for virtual knots which can be taken from [1, 2], would, if generalized for virtual knot theory, take values in these graphs.
2. The graphical information can say a lot about a knot: all irreducible (no bigons and no quadrilaterals) graphs which appear in the value of the invariant should appear “inside” any diagram of the knot, which, in turn, says a lot about the knot itself, in particular, about its minimality, and non-classicality.

The present paper is organized as follows. Section 2 is a review of concepts from virtual knot theory, flat virtual knot theory and free knot theory. Section 3 discusses parity in virtual knot theory, starting with the odd writhe of a virtual knot and then working with the parity bracket for free knots. Section 4 contains the construction of the main invariant in this paper, generalizing the Kuperberg bracket for $sl(3)$. Section 5 gives minimality results related to our invariant. Section 6 reformulates our work for braids and shows that the extension of the Kuperberg $sl(3)$ invariant gives a trace on the virtual Hecke algebra. The classical Hecke algebra is a quotient of the group ring of the Artin braid group (coefficients in a given commutative ring) by the skein relation for the Homflypt polynomial written in braid form as

$$\sigma_i - \sigma_i^{-1} = z1$$

where σ_i denotes a generator of the braid group and z is a chosen algebra element. In the Hecke algebra the braiding generators satisfy a quadratic relation and one can use this relation to create a trace function on the Hecke algebra that reproduces the Homflypt polynomial. In particular, this trace can reproduce the classical Kuperberg bracket. We show how our graphical extensions

can continue this structure for virtual braids with a trace that is valued in graph polynomials. Section 7 shows how our invariant generalizes a bracket of R. Penrose that counts colorings of plane graphs and raises questions about an unoriented reductive invariant related directly to the Penrose bracket. We end by showing a direct relationship between binary edge colorings of free links and colorings of trivalent graphs.

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2 Basics of Virtual Knot Theory

2.1 Virtual Knot Theory

Knot theory studies the embeddings of curves in three-dimensional space. Virtual knot theory studies the embeddings of curves in thickened surfaces of arbitrary genus, up to the addition and removal of empty handles from the surface. Virtual knots have a special diagrammatic theory, described below, that makes handling them very similar to the handling of classical knot diagrams. Many structures in classical knot theory generalize to the virtual domain.

In the diagrammatic theory of virtual knots one adds a *virtual crossing* (see Figure 3) that is neither an over-crossing nor an under-crossing. A virtual crossing is represented by two crossing segments with a small circle placed around the crossing point.

Note that a classical knot vertex is a 4-valent graphical node embedded in the plane with extra structure. The extra structure includes the diagrammatic choice of crossing (indicated by a broken segment) and a specific choice of cyclic order (counterclockwise when embedded in the plane) at the vertex. A virtual knot is completely specified by its 4-valent nodes if the edges incident to the nodes are labeled so that they can be connected by arcs to form the corresponding graph.

A virtual diagram is an immersion of a collection of circles into the plane such that some crossings are structured as classical crossings and some are simply labeled as virtual crossings and indicated by a small circle drawn around the crossing. We regard the resulting diagram as a possible non-planar graph whose only nodes are the classical crossings, with their cyclic structure. Any immersion of such a graph, preserving the cyclic structure at the nodes, will represent the *same* virtual knot or link. From this, we postulate the *detour move* (see below) for arcs with consecutive virtual crossings, so that this equivalence is satisfied. The reader used to working with graphs will note that our virtual crossings are direct analogs to the extra crossings of edges that occur when non-planar graphs are drawn in the plane.

Moves on virtual diagrams generalize the Reidemeister moves for classical knot and link diagrams (Figure 3). One can summarize the moves on virtual diagrams by saying that the classical crossings interact with one another according to the usual Reidemeister moves while virtual crossings are artifacts of the attempt to draw the virtual structure in the plane. A segment of diagram

consisting of a sequence of consecutive virtual crossings can be excised and a new connection made between the resulting free ends. If the new connecting segment intersects the remaining diagram (transversally) then each new intersection is taken to be virtual. Such an excision and reconnection is called a *detour move*. Adding the global detour move to the Reidemeister moves completes the description of moves on virtual diagrams. In Figure 3 we illustrate a set of local moves involving virtual crossings. The global detour move is a consequence of moves (B) and (C) in Figure 3. The detour move is illustrated in Figure 4. Virtual knot and link diagrams that can be connected by a finite sequence of these moves are said to be *equivalent* or *virtually isotopic*. A virtual knot is an equivalence class of virtual diagrams under these moves.

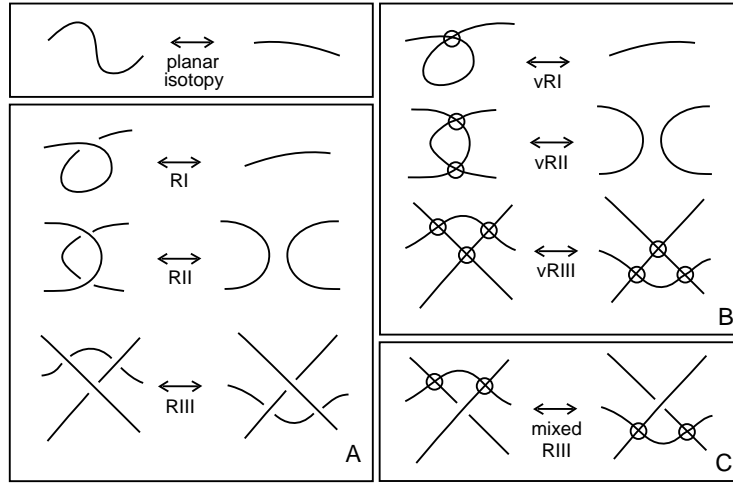


Figure 3: **Moves**

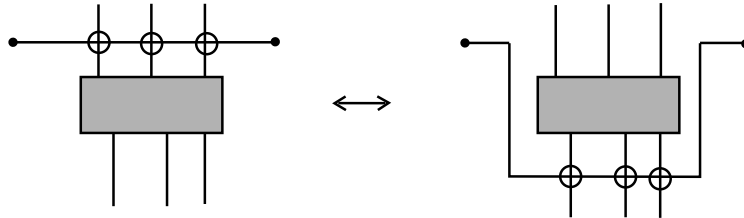


Figure 4: **Detour Move**

Another way to understand virtual diagrams is to regard them as representatives for oriented Gauss codes [7], [17, 18] (Gauss diagrams). Such codes do not always have planar realizations. An attempt to embed such a code in the plane leads to the production of the virtual crossings. Gauss codes and diagrams are most convenient for knots, where there is one cycle in the code and one circle in the Gauss diagram. We can work with Gauss diagrams for links, but it requires extra care. The detour move makes the particular choice of virtual crossings irrelevant. *Virtual isotopy is the same as the equivalence relation generated on the collection of oriented Gauss codes*

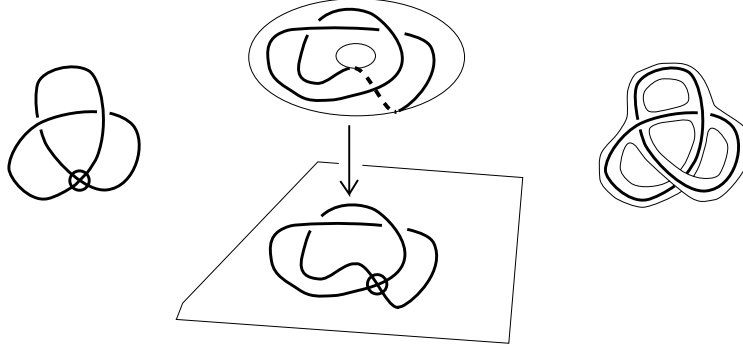


Figure 5: **Surfaces and Virtuals**

by abstract Reidemeister moves on these codes. We omit detailed discussion of Gauss diagrams here. The reader can see this approach in [15, 7, 28]. It is of interest to know the least number of virtual crossings that can occur in a diagram of a virtual knot or link. If this virtual crossing number is zero, then the link is classical. For some results about estimating virtual crossing number see [9, 22, 30].

2.2 Interpretation of Virtual Links as Stable Classes of Links in Thickened Surfaces

There is a topological interpretation [17, 19] for the virtual theory in terms of embeddings of links in thickened surfaces. Regard each virtual crossing as a shorthand for a detour of one of the arcs in the crossing through a 1-handle that has been attached to the 2-sphere of the original diagram. By interpreting each virtual crossing in this way, we obtain an embedding of a collection of circles into a thickened surface $S_g \times \mathbb{R}$ where g is the number of virtual crossings in the original diagram L , S_g is a compact, connected, oriented surface of genus g and \mathbb{R} denotes the real line. We say that two such surface embeddings are *stably equivalent* if one can be obtained from another by isotopy in the thickened surfaces, homeomorphisms of the surfaces and handle stabilization.

To define handle stabilization, regard the knot or link as represented by a diagram D on a surface S . If C is an embedded curve in S that does not intersect the diagram D and cutting along D does not disconnect the surface, then we cut along C and add two disks to fill in the boundary of the cut surface. This is a handle stabilization move that reduces the genus of the surface to a surface S' containing a new diagram D' . The pairs (S, D) and (S', D') represent the same virtual knot or link. The reverse operation that takes (S', D') to (S, D) consists in choosing two disks in S' that are disjoint from D' , cutting them out and joining their boundaries by a tube (hence the term handle addition for this direction of stabilization).

We have the

Theorem 1 [17, 15, 19, 8]. *Two virtual link diagrams are isotopic if and only if their corresponding surface embeddings are stably equivalent.*

In Figure 5 we illustrate some points about this association of virtual diagrams and knot diagrams on surfaces. Note the projection of the knot diagram on the torus to a diagram in the plane (in

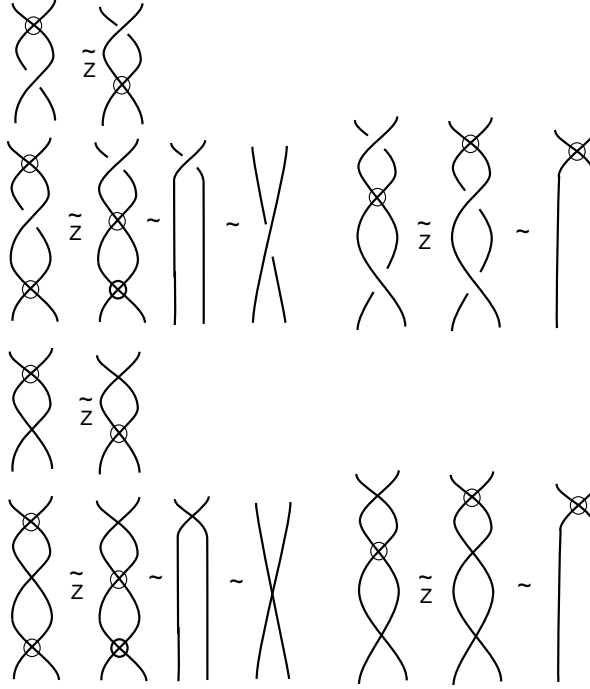


Figure 6: **The Z-Move**

the center of the figure) has a virtual crossing, an artifact in the planar diagram, where two arcs that do not form a crossing in the thickened surface project to the same point in the plane. In this way, virtual crossings can be regarded as artifacts of projection. The same figure shows a virtual diagram on the left and an “abstract knot diagram” [14, 8] on the right. The abstract knot diagram is a realization of the knot on the left in a thickened surface with boundary and it is obtained by making a neighborhood of the virtual diagram that resolves the virtual crossing into arcs that travel on separate bands. The virtual crossing appears as an artifact of the projection of this surface to the plane. Note that in the neighborhood of a crossing we always project so that the orientation of the surface agrees with the orientation of the projection plane. To systematize the association of virtual knot diagram to diagram in a surface, construct the abstract knot diagram by taking a neighborhood of the diagram in the surface; then arrange the abstract diagram so that all positive surface normals point upwards, and project it to the plane. The reader will find more information about this correspondence [17, 15] in other papers by the authors and in the literature of virtual knot theory.

2.3 Flat and Free Knots and Links

As we have said, every classical knot diagram can be regarded as a 4-regular plane graph with extra structure at the nodes. If we take the flat diagram without this extra structure, then the diagram is the shadow of some link in three dimensional space, but the weaving of that link is not specified. It is well known that if one is allowed to apply the Reidemeister moves to such a classical shadow then the shadow can be reduced to a disjoint union of circles. This reduction is

no longer true for virtual links. More precisely, let a *flat virtual diagram* be a diagram with *virtual crossings* as we have described them and *flat crossings* consisting in undecorated nodes of the 4-regular plane graph, retaining the cyclic order at a node. Two flat virtual diagrams are *equivalent* if there is a sequence of generalized flat Reidemeister moves (as illustrated in Figure 3) taking one to the other. A generalized flat Reidemeister move is any move as shown in Figure 3 where one ignores the over or under crossing structure. The reader can obtain for himself an illustration of the moves for flat virtual knots by taking Figure 3 and replacing all the classical crossings by flat (but not virtual) crossings. Note that in studying flat virtuals the rules for changing virtual crossings among themselves and the rules for changing flat crossings among themselves are identical. But detour moves as in part C of Figure 3 are available for virtual crossings with respect to flat crossings and *not* the other way around.

We shall say that a virtual diagram *overlies* a flat diagram if the virtual diagram is obtained from the flat diagram by choosing a crossing type for each flat crossing in the virtual diagram. To each virtual diagram K there is an associated flat diagram $F(K)$, obtained by forgetting the extra structure at the classical crossings in K . Note that if K and K' are isotopic as virtual diagrams, then $F(K)$ and $F(K')$ are isotopic as flat virtual diagrams. Thus, if we can show that $F(K)$ is not reducible to a disjoint union of circles, then it will follow that K is a non-trivial and non-classical virtual link.

The theory of flat virtual knots and links can also be formulated in terms of Gauss codes or Gauss diagrams. We will not describe this coding method here. In Gauss diagrams there are no virtual crossings. Virtual crossings are an artifact of the realization of the flat diagram in the plane. In Turaev's work [32] flat virtual knots and links are called *virtual strings*. See also recent papers of Roger Fenn [11, 12] for other points of view about flat virtual knots and links. In particular see [13] and [28] for two points of view that give a complete classification of flat virtual knots.

Definition. A virtual graph is a 4-regular graph that is immersed in the plane giving a choice of cyclic orders at its nodes. The edges at the nodes are connected according to the abstract definition of the graph and are embedded into the plane so that they intersect transversely. These intersections are taken as virtual crossings and are subject to the detour move just as in virtual link diagrams. Two virtual graphs are *isotopic* if there is a combination of planar graph isotopies and detour moves that connect them. A *virtual graph* is regarded as a flat virtual diagram where the classical flat crossings are not subjected to the flat Reidemeister moves. The theory of virtual graphs is equivalent to the theory of 4-regular graph diagrams embedded in oriented surfaces, taken up to handle stabilization.

Framed Nodes and Framed Graphs. We shall also use the concept of a *framed 4-valent node* where we only specify the pairings of *opposite edges* at the node. In the cyclic order, two edges are said to be opposite if they are paired by skipping one edge as one goes around. If the cyclic order of a node is $[a, b, c, d]$ where these letters label the edges incident to the node, then we say that edges a and c are *opposite*, and that edges b and d are it opposite. We can change the cyclic order and keep the opposite relation. For example, in $[c, b, a, d]$ it is still the case that the opposite pairs are a, c and b, d . A *framed 4-valent graph* is a 4-valent graph where every node is framed. When we represent a framed 4-valent graph as an immersion in the plane, we use virtual crossings for the edge-crossings that are artifacts of the immersion and we regard the graph as a virtual graph. For a general framed 4-valent graph, no Reidemeister moves are allowed except for

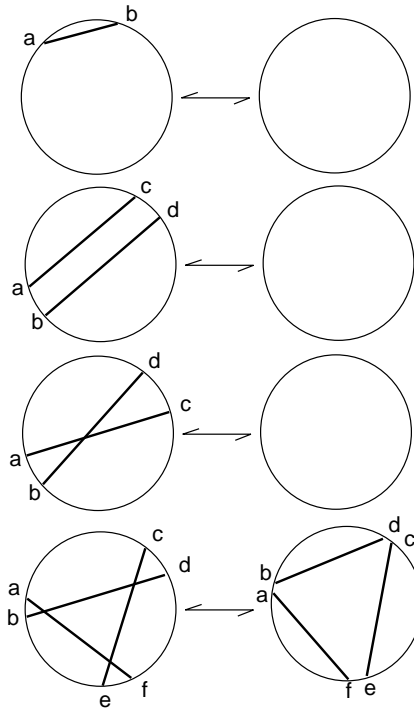


Figure 7: Reidemeister Moves on Chord Diagrams

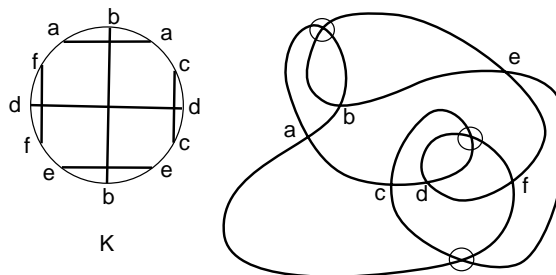


Figure 8: The Free Knot K - Chord Diagram and Virtual Diagram

those among the virtual crossings.

It should be remarked that due to the relationships with knot theory, we usually allow circles along with the graphs of any kind in our work with graph theory. It will often go without saying that circles are part of the discussion.

We also need to remark about the notion of a *component* of a framed graph. A component is obtained by taking a walk on the graph so that the walk contains pairs of opposite edges from every node that is met during the walk. That is, in walking, if you enter a node along a given edge, then you exit the node along its opposite edge. Such a walk produces a cycle in the graph and such cycles are called the *components* of the framed graph. Since a link diagram or a flat link diagram is a framed graph, we see that the components of this framed graph are identical with the components of the link as identified by the topologist. A framed graph with one component is said to be *unicursal*. We will use this terminology in Section 4 of the present paper.

When we take virtual knot diagrams only up to framing of their classical nodes, we are allowing the Z – move as illustrated in Figure 6. In the Z – move one can interchange a crossing and an adjacent virtual crossing. If the non-virtual crossing is a classical crossing then it retains its crossing type as shown in the figure. In a Gauss diagram it would retain its arrow but reverse its sign. For a flat crossing, there is nothing further to say. We call virtual knots and links modulo the Z -move *Z -knots*, and we call flat virtual knots modulo the Z -move *free knots*. Thus free knots are the same as *framed 4-valent graphs taken up to the flat Reidemeister moves. This is the same as arbitrary Gauss diagrams without sign or arrow, taken up to the Reidemeister moves.*

Free knots and links are represented by chord diagrams with no signs or arrows on the chords. These chord diagrams are subject to the usual abstract Reidemeister moves without any restrictions on signs. See Figure 7 for an indication of these moves. See Figure 8 for a Gauss diagram and corresponding planar diagram with virtual crossings for a free knot K that turns out to be non-trivial. The non-triviality of this free knot follows from the fact that all the chords in its chord diagram are *odd* (they have an odd number of intersections with the other chords) and the use of the Manturov Parity Bracket invariant [27]. The theory of free knots is identical with the theory that one gets when one takes Flat Virtuals modulo the flat Z -move as shown in Figure 6. We say that a *virtualization move* has been performed on a crossing if it is flanked by two virtual crossings. We illustrate this operation in Figure 6 and show that virtualization does not change the equivalence class of a flat diagram under the Z -move. This means that any invariant of free knots must be invariant under virtualization.

3 Parity in Knot Theory and Virtual Knot Theory

This section discusses the use of parity in knot theory, virtual knot theory and particularly in the theory of free knots. See [28, 27, 29] for recent work in this area.

3.1 The Odd Writhe

Using the Odd Writhe $J(K)$. In Figure 9 we show that the virtual knot K is not trivial, not classical and not equivalent to its mirror image by computing its *odd writhe* [21]. The odd writhe [21], $J(K)$, is the sum of the signs of the odd crossings. A crossing in a knot diagram is *odd* if it flanks an odd number of symbols in the Gauss code of the diagram. We call this *Gaussian parity*

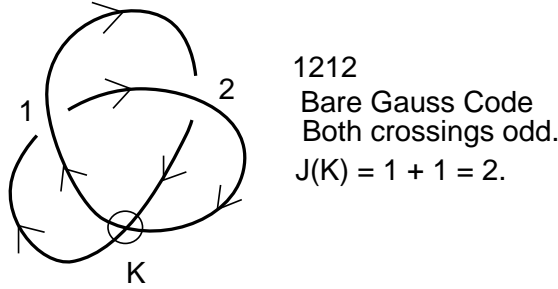


Figure 9: **Example of Odd Writhe, $J(K)$.**

to distinguish it from other parities that can be defined for virtual diagrams. All crossings in a classical knot diagram are even. Classical diagrams have zero odd writhe. In the figure, the flat Gauss code for K is 1212 with both crossings odd. Thus we see that $J(K) = 2$ for the knot in the figure. One proves that

1. $J(K)$ is an isotopy invariant of virtual knots,
2. $J(K^*) = -J(K)$ when K^* is the mirror image of K (obtained by switching all the crossings),
3. $J(K) = 0$ when K is isotopic to a classical knot.

The odd writhe is the simplest application of parity in virtual knot theory. In the next section we show how to use parity to detect certain free knots.

3.2 The Parity Bracket

In this section we introduce the Parity Bracket Invariant of Vassily Manturov [27]. This is a form of the bracket polynomial for free knots that uses the parity of the crossings. Some terminology will be useful: We refer to the *graph* of a free knot as the intersection graph of its chord diagram. That is, the nodes of the graph are in $1-1$ correspondence with the chords of the chord diagram, and two nodes are joined by an edge when the two chords intersect in the chord diagram. If a graph is the intersection graph for the chord diagram of a free knot, we say that the free knot is *generated* by the graph. The reader should note that intersection graphs have neither loops nor bigons. In Figure 11 we illustrate the relationship of an intersection graph and its chord diagram.

We call a free knot diagram *odd* if all of the nodes in its graph have odd degree. (This is the same as saying that every crossing has odd Gaussian parity). We call a simple (no more than one edge between any two nodes) graph *irreducibly odd* if it is odd and for every pair of nodes u and v there is a third node w that is adjacent to exactly one of u and v .

The simplest example of an irreducibly odd graph is depicted in Fig. 11. Assume an irreducibly odd graph G generates a free knot K . It turns out that the representative G of the knot K is indeed *minimal*: any other representative of K has the number of vertices at least as many as those of G . To this end, we shall describe a powerful invariant [27] that captures graphical information about the knot; but first we should introduce some notation.

Let \mathfrak{G} be the set of all equivalence classes of framed graphs with one unicursal component modulo second Reidemeister moves. Consider the linear space $\mathbb{Z}_2\mathfrak{G}$. Let G be a framed graph, let

$$\begin{aligned}
\langle \text{diag}_1 \rangle_P &= \langle \text{diag}_2 \rangle_P + \langle \text{diag}_3 \rangle_P \\
\langle \text{diag}_4 \rangle_P &= \langle \text{diag}_5 \rangle_P \\
\langle \text{diag}_6 \rangle_P &= \langle \text{diag}_7 \rangle_P \\
\langle \text{diag}_8 \rangle_P &= 0
\end{aligned}$$

Figure 10: Parity Bracket Expansion

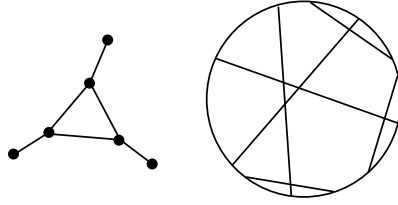


Figure 11: An irreducibly odd graph and its chord diagram

v be a vertex of G with four incident half-edges a, b, c, d , s.t. a is opposite to c and b is opposite to d at v . By *smoothing* of G at v we mean any of the two framed 4-graphs obtained by removing v and repasting the edges as $a - b, c - d$ or as $a - d, b - c$, see Fig. 12.

Herewith, the rest of the graph (together with all framings at vertices except v) remains unchanged. We may then consider further smoothings of G at *several* vertices. Consider the following sum

$$[G] = \sum_{s \text{ even}, 1 \text{ comp}} G_s, \quad (1)$$

which is taken over all smoothings in all *even* vertices, and only those summands are taken into

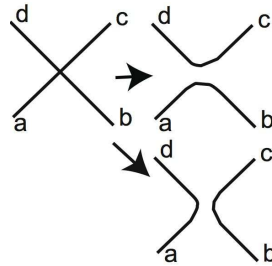


Figure 12: Two smoothings of a vertex of for a framed graph

account where G_s has one unicursal component.

Thus, if G has k even vertices, then $[G]$ will contain at most 2^k summands, and if all vertices of G are odd, then we shall have exactly one summand, the graph G itself. Consider $[G]$ as an element of $\mathbb{Z}_2\mathfrak{G}$. In this case it is evident that if all vertices of G are odd then $[G] = G$ (here G is taken as a single graph to be reduced by R-2 moves).

Theorem 1. *If G and G' represent the same free knot then in $\mathbb{Z}_2\mathfrak{G}$ the following equality holds: $[G] = [G']$.*

Theorem 1 yields the following

Corollary 1. *Let G be an irreducibly odd framed 4-graph with one unicursal component. Then any representative G' of the free knot K_G , generated by G , has a smoothing \tilde{G} having the same number of vertices as G . In particular, G is a minimal representative of the free knot K_G with respect to the number of vertices.*

It turns out that elements from $\mathbb{Z}_2\mathfrak{G}$ are easily encoded by their minimal representatives. More precisely, the following lemma holds.

Lemma 1. *Every 4-valent framed graph G with one unicursal component considered as an element of $\mathbb{Z}_2\mathfrak{G}$ has a unique irreducible representative, which can be obtained from G by consecutive application of second decreasing Reidemeister moves.*

This allows one to recognize elements $\mathbb{Z}_2\mathfrak{G}$ easily, which makes the invariants constructed in the previous subsection digestable. In particular, the minimality of a framed 4-graph in $\mathbb{Z}_2\mathfrak{G}$ is easily detectable: one should just check all pairs of vertices and see whether any of them can be cancelled by a second Reidemeister move (or in $\mathbb{Z}_2\mathfrak{G}$ one should also look for free loops). Create all minimal representatives in this way, and then compare them. Indeed, we see that two graphs are R-2-equivalent if their minimal representatives coincide.

Proof of the Corollary. By definition of $[G]$ we have $[G] = G$. Thus if G' generates the same free knot as G we have $[G'] = G$ in $\mathbb{Z}_2\mathfrak{G}$. Consequently, the sum representing $[G']$ in \mathfrak{G} contains at least one summand which is a -equivalent (a -equivalent means equivalent by Reidemeister two-moves only) to G . Thus G' has at least as many vertices as G does. Moreover, the corresponding smoothing of G' is a diagram, which is a -equivalent to G . One can show that under some (quite natural) “rigidity” condition this will yield that one of smoothings of G' coincides with G . \square

The use of parity gives this bracket considerable power. For example, consider the diagram in Figure 8. We see that all the crossings in this free knot are odd. Thus the parity bracket is just the diagram itself seen as a graph and reduced by 2-moves. The reader can check that this graph is in fact irreducible and the invariant of this free knot is the diagram itself as an irreducible graph. This not only means that this free knot is non-trivial, it also means that any diagram of this free knot will have states that reduce to the diagram in Figure 8.

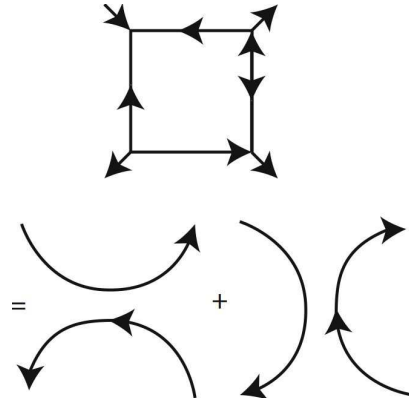
Parity is clearly an important theme in virtual knot theory and will figure in many future investigations of this subject. The type of construction that we have indicated for the bracket polynomial in this section can be varied and applied to other invariants. Furthermore the notion of describing a parity for crossings in a diagram is also susceptible to generalization. For more on this theme the reader should consult [27, 29, 23] and [21] for a use of parity for another variant of the bracket polynomial.

4 Construction of the Main Invariant

Let \mathcal{S} be the collection of all trivalent bipartite graphs with edges oriented from vertices of the first part to vertices of the second part. Let $\mathcal{T} = \{t_1, t_2, \dots\}$ be the (infinite) subset of graphs from \mathcal{S} having neither bigons nor quadrilaterals. Let \mathcal{M} be the module $\mathbb{Z}[A, A^{-1}][t_1, t_2, \dots]$ of formal commutative products of graphs from \mathcal{T} with coefficients that are Laurent polynomials. Our main invariant will be valued in \mathcal{T} .

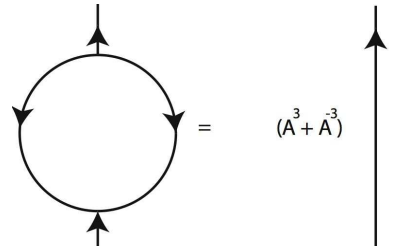
Statement 1. *There exists a unique map $f : \mathcal{T} \rightarrow \mathcal{M}$ which satisfies the relations in Figure 2 and the further relations shown below in (2), (3) and (4). The entire set of relations is summarized in Figure 19.*

Proof. The relations we are going to use to prove the statement are the following:



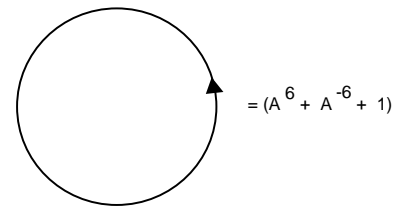
(2)

Resolving a square



(3)

Resolving a bigon



(4)

The value of a loop

Note that for the case of planar tangles this map was constructed explicitly by Kuperberg [3], and the image was in $\mathbb{Z}[A, A^{-1}]$. We are going to follow [3], however, in the non-planar case, the graphs can not be reduced just to collections of closed curves (in the case of the plane,

Jordan curves) which later evaluate to numbers. In fact, irreducible graphs will appear in the non-planar case. First, we treat every 1-complex with all components being graphs from \mathcal{T} and circles: We treat it as the formal product of these graphs, where each circle evaluates to the factor $(A^6 + A^{-6} + 1)$. We note that if a graph Γ from \mathcal{T} has a bigon or a quadrilateral, then we can use the relations (2,3,4) to reduce it to a smaller graph (or two graphs, then we consider it as a product). So, we can proceed with resolving bigons and quadrilaterals until we are left with a collection of graphs t_j and circles; this gives us an element from \mathcal{M} ; once we prove the uniqueness of the resolution, we set the stage for proving the existence of the invariant. We must carefully check well-definedness and topological invariance.

In what follows, we shall often omit the letter f by identifying graphs with their images or intermediate graphs which appear after some concrete resolutions.

Our goal is to show that this map $f : \mathcal{T} \rightarrow \mathcal{M}$ is well-defined. We shall prove it by induction on the number of graph edges. *The induction base is obvious* and we leave its articulation to the reader. To perform the induction step, notice that all of Kuperberg's relations are reductive: from a graph we get to a collection of simpler graphs.

Assume for all graphs with at most $2n$ vertices that the statement holds. Now, let us take a graph Γ from \mathcal{T} with $2n + 2$ vertices. Without loss of generality, we assume this graph is connected. If it has neither bigon nor quadrilateral, we just take the graph itself to be its image. Otherwise we use the relations(3) or (2) to reduce it to a linear combination of simpler graphs; we proceed until we have a sum (with Laurent polynomial coefficients) of (products of) graphs without bigons and quadrilaterals.

According to the induction hypothesis, for all simpler graphs, there is a unique map to \mathcal{M} . However, we can apply the relations in different ways by starting from a given quadrilateral or a bigon. We will show that the final result does not depend on the bigon or quadrilateral we start with. To this end, it suffices to prove that if Γ can be resolved to $\alpha\Gamma_1 + \beta\Gamma_2$ from one bigon (quadrilateral) and also to $\alpha'\Gamma'_1 + \beta'\Gamma'_2$ from the other one, then both linear combinations can be resolved further, and will lead to the same element of \mathcal{T} . This will show that final reductions are unique.

Whenever two nodes of a quadrilateral coincide, then two edges coincide and it is no longer subject to the quadrilateral reduction relation. Thus we assume that quadrilaterals under discussion have distinct nodes. Note that if two polygons (bigons or quadrilaterals) share no common vertex then the corresponding two resolutions can be performed *independently* and, hence, the result of applying them in any order is the same. So, in this case, $\alpha\Gamma_1 + \beta\Gamma_2$ and $\alpha'\Gamma'_1 + \beta'\Gamma'_2$ can be resolved to the same linear combination in one step. By the hypothesis, $f(\Gamma_1), f(\Gamma_2), f(\Gamma'_1), f(\Gamma'_2)$ are all well defined, so, we can simplify the common resolution for $\alpha\Gamma_1 + \beta\Gamma_2$ and $\alpha'\Gamma'_1 + \beta'\Gamma'_2$ to obtain the correct value for f of any of these two linear combinations, which means that they coincide.

If two polygons (bigons or quadrilaterals) share a vertex, then they share an edge because the graph is trivalent. If a connected trivalent graph has two different bigons sharing an edge then the total number of edges of this graph is three, and the evaluation of this graph in \mathcal{T} follows from an easy calculation. Therefore, let us assume we have a graph Γ with an edge shared by a bigon and a quadrilateral. We can resolve the quadrilateral first, or we can resolve the bigon first. The calculation in Fig. 13 shows that after a two-step resolution we get to the same linear combination of a graph.

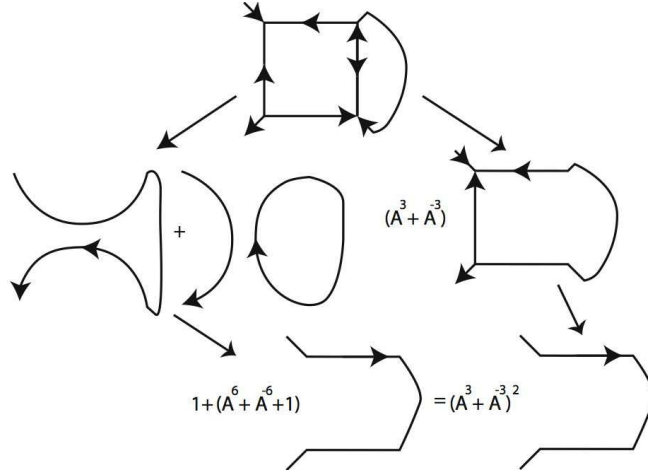


Figure 13: Two Ways of Reducing a Quadrilateral and a Bigon

A similar situation happens when we deal with two quadrilaterals sharing an edge, see Fig. 14. Here we have shown just one particular resolution, but the picture is symmetric, so the result of the resolution when we start with the right quadrilateral, will lead us to the same result. See also Fig. 15 and Fig. 16. These figures illustrate two other ways in which the edge can be shared. Note that Fig. 15 illustrates a possibly non-planar case, and that we use the abstract graph structure (no particular order at the trivalent vertex) in the course of the evaluation. These cases cover all the ways that shared edges can occur, as the reader can easily verify.

Thus, we have performed the induction step and proved the well-definedness of the mapping. Note that the ideas of the proof are the same as in the case of classical case; however, we never assumed any planarity of the graph; we just draw graphs planar whenever possible. Note that the situation in Figure 15 is principally non-planar. The invariance under virtualization follows from this definition because the graphical pieces into which we expand a crossing, as in Figure 1, are, as graphs, symmetric under the interchange produced by the virtualization. \square

Remark. We can, in the case of flat knots or standard virtual knots represented on surfaces, enhance the invariant by keeping track of the embedding of the graph in the surface and only expanding on bigons and quadrilaterals that bound in the surface. We will not pursue this version of the invariant here. In undertaking this program we will produce evaluations that are not invariant under the Z -move for flats or for virtual knots.

Now we give a formal description of our main invariant. This evaluation is invariant under the Z – move. It is defined for virtual knots and links and it specializes to an invariant of free knots. Let K be an oriented virtual link diagram. With every classical crossing of K , we associate two local states: the oriented one and the unoriented one: the oriented one shown in the upper picture of Fig. 2, and the unoriented one shown in the lower picture of Fig. 2. A *state* of the diagram is a choice of local state for each crossing in the diagram.

We define the bracket $[[\cdot]]$ (generalized Kuperberg bracket) as follows. Let K be an oriented virtual link diagram. For a state s of a virtual knot diagram K , we define the weight of the state as the coefficient of the corresponding graph according to the Kuperberg relations (2). More

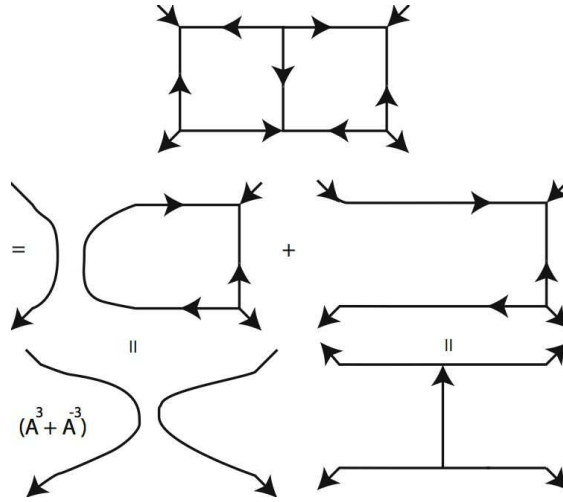


Figure 14: **Resolving Two Adjacent Squares**

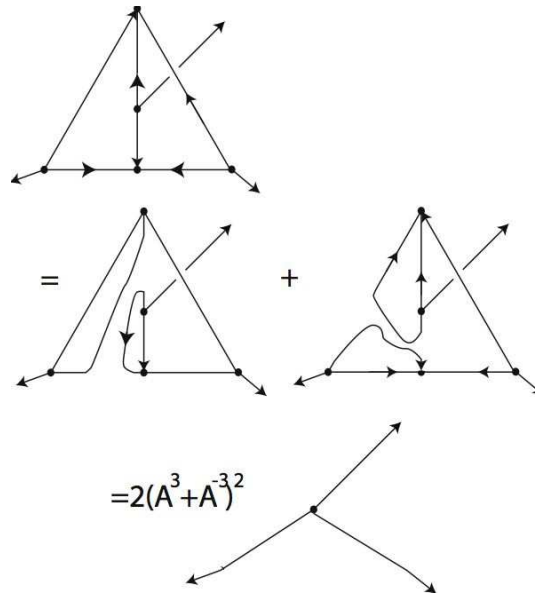


Figure 15: **Resolving Two Different Adjacent Squares**

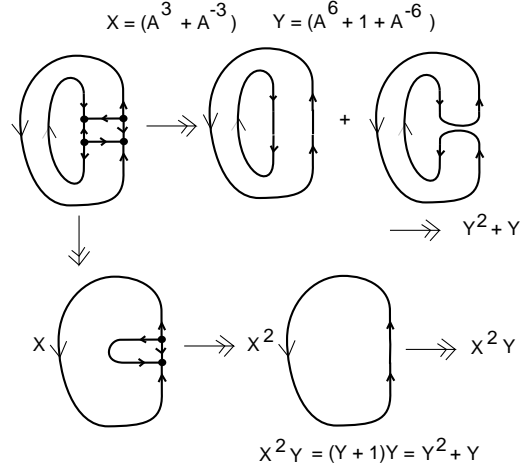


Figure 16: **Resolving Two Annular Squares**

precisely, the weight of a state is the product of weights of all crossings, where a weight of a positive crossing is A^{2wr} for the oriented resolution and $-A^{-wr}$ for the unoriented resolution, wr stands for the writhe number (the oriented sign) of the crossing.

Set

$$\sum_s w(K_s) \cdot f(K_s) \in \mathcal{M}, \quad (5)$$

where $w(s)$ is the weight of the state.

Theorem 2. *The bracket $[[\cdot]]$ is invariant under all Reidemeister moves and the virtualization move.*

Proof. The invariance proof under Reidemeister moves repeats that of Kuperberg. The only thing we require is that the application of the Kuperberg relations (summarized in Figure 19) can be applied to yield unique reduced graph polynomials. the discussion preceding the proof handles this issue. \square

From the definition of $[[K]]$ we have the following

Corollary 2. *If $[[K]]$ does not belong to $\mathbb{Z}[A, A^{-1}] \subset \mathcal{M}$ then the knot $[[K]]$ is not classical.*

Recalling that a free link is an equivalence of virtual knots modulo virtualizations and crossing switches and taking into account that the skein relations 2 for $[[\cdot]]$ for \times and \times are the same when specifying $A = 1$, we get the following

Corollary 3. *$[[K]]_{A=1}$ and $[[K]]_{A=-1}$ are invariants of free links.*

By the unoriented state K_{us} of virtual knot diagram (resp., free knot diagram) K we mean the state of K where all crossings are resolved in a way where an edge is added. **Notation:** K_{us} . Note that K_{us} is treated as a graph.

Corollary 4. *Assume for a virtual knot (or free knot) K with n classical crossings the graph K_{us} has neither bigons nor quadrilaterals. Then every knot K' equivalent to K has a state s such that K'_s contains K_{us} as a subgraph. This state can be treated as an element of \mathcal{S} . In particular, K is minimal, and all minimal diagrams of this free knot have the same number of crossings.*

Note that the coincidence of K_{us} and K'_{us} does not guarantee the coincidence of K and K' . For example, if K and K' differ by a third unoriented Reidemeister move, then, of course, $[[K]] = [[K']]$. The corresponding resolutions K_{us} and K'_{us} will coincide (they will have a hexagon inside). See the Minimality Section for more information.

Corollary 5. *Let K be a four-valent framed graph with n crossings and with girth number at least five. Then the hypothesis of Corollary 4 holds.*

So, this proves the minimality of a large class of framed four-valent graphs regarded as free knots: all graphs having girth ≥ 5 and many other knots. For example, consider the free knot K_n whose Gauss diagram is the n -gon, $n > 6$: it consists of n chords where i -th chord is linked with exactly two chords, those having numbers $i - 1$ and $i + 1$ (the numbers are taken modulo n). Then K_n satisfies the condition of 4 and, hence, is minimal in a strong sense.

Note that the triviality of such n -gons as free knots was proven only for $n \leq 6$.

Remark 1. The above argument works for links and tangles as well as knots.

From the construction of $[[\cdot]]$ we get the following corollary.

Corollary 6. *Let K be a virtual (resp., flat) knot, and let $\Gamma_1 \cdots \Gamma_k$ be a product of irreducible graphs which appear as a summand in $[[K]]$ (resp., $[[K]]|_{A=1}$) with a non-zero coefficient. Then the minimal virtual crossing number of K is greater than or equal to the sum of crossing numbers of graphs: $cr(\Gamma_1) + \cdots + cr(\Gamma_k)$ and the underlying genus of K is less than or equal to the sum of genera $g(\Gamma_1) + \cdots + g(\Gamma_k)$ (in virtual or free knot category).*

The above corollary easily allows one to reprove the theorem first proved in [30], that the number of virtual crossings of a virtual knot grows quadratically with respect to the number of classical crossings for some families of graphs. In [30], it was done by using the parity bracket. Now, we can do the same by using

$$Free[[K]] = [[K]]|_{A=1}.$$

With this invariant one can easily construct infinite series of trivalent bipartite graphs which serve as K_{us} for some sequence of knots K_n and such that the minimal crossing number for these graphs grows quadratically with respect to the number of vertices. Recalling that the number of vertices comes from the number of classical crossings of K_n , we get the desired result.

5 Minimality Theorems and Uniqueness of Minimal Diagrams

The existence of non-trivial free knots was first proved in [6]. The proof relied upon the notion of *parity*: the way of discriminating between *even* and *odd* crossings of the knot diagram (respectively, chords of the chord diagram). The simplest example of parity is *the Gaussian parity* as described in the second section of this paper. In the third section of this paper, we have discussed Corollary 1 showing that an irreducible odd diagram of a free knot K is minimal in the strong sense that every diagram K' equivalent to K has a smoothing identical with K .

Here a diagram is called *odd* if all crossings of it are odd, and a diagram is *irreducible* if it no decreasing second Reidemeister move can be applied to it; from the point of view of Gauss diagrams this means that there are no two parallel or adjacent chords as shown in Figure 7. This Corollary 1 is a strong statement: for example, it follows from it that the minimal diagram is

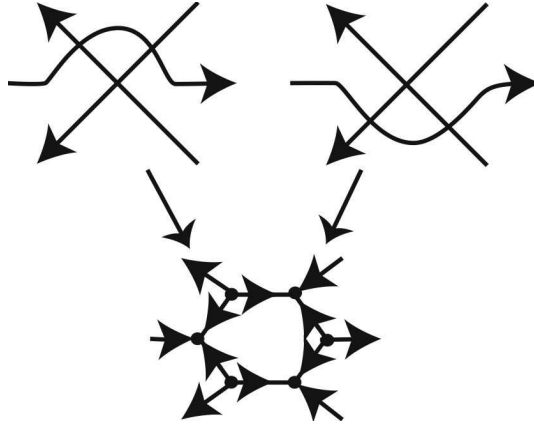


Figure 17: **The third move does not change the bracket**

unique: if K is an n -crossing diagram, then no other n -crossing diagram K' can have K as a smoothing since non-trivial smoothing leads to a smaller number of classical crossings.

This result partially solves the recognition problem for free knots, but it is solved for a very small class of them. Indeed, there are lots of minimal diagrams of free knots which are minimal and have even crossings.

The results we have proved above (Corollaries 4 and 5) have this level of strength. Indeed, if we take $Free[[K]] = [[K]]|_{A=1}$ for a flat knot, then if K_{us} has neither bigons nor loops then K is minimal, and *this may happen for diagrams with all even crossings*. Here we see one of the features that is new in using our generalization of the Kuperberg bracket.

The result stated in Corollaries 4 and 5 does not claim the uniqueness of the minimal diagram. The main reason is that we can not restore the initial knot (or link) K from K_{us} in a unique way. Indeed, given a trivalent graph, in order to get a framed four-valent graph, we have to contract some edges (those corresponding to unoriented smoothings of vertices). This means that we have to find a matching for the set of vertices of the whole trivalent graph K_{us} . But there might be many matchings of this sort. After performing such a matching, we still do not have a free knot diagram (framed four-valent diagram), because we do not know which incoming edge is opposite to which emanating edge. Thus, there may be many diagrams K having the same K_{us} . Of course, in some cases, the whole invariant $Free[[K]]$ itself does distinguish between them, nevertheless, sometimes it is not the case. View Fig. 17. The bracket $K \mapsto Flat[[K]]$ is certainly invariant under any Reidemeister moves; so is K_{us} in the case of Corollary 4.

Now, we see that in Fig. 17, two diagrams which differ by a third Reidemeister move, lead to the same K_{us} . However, we do not know whether there are different minimal knot diagrams of *different knot types* sharing K_{us} (or, even, sharing $Free[[K]]$). If the unoriented third Reidemeister move were the only case when $Free[[K]]$ does not change then we would have not only a criterion for minimality of a large class of diagrams, but also a recognition algorithm for a large class of free knots.

A free knot diagram (a framed four-valent graph) can not contain loops or bigons. If a free knot diagram K contains a triangle, then it is equivalent to a diagram K' obtained from K by a third Reidemeister move. If K has girth greater than or equal to four, then no decreasing

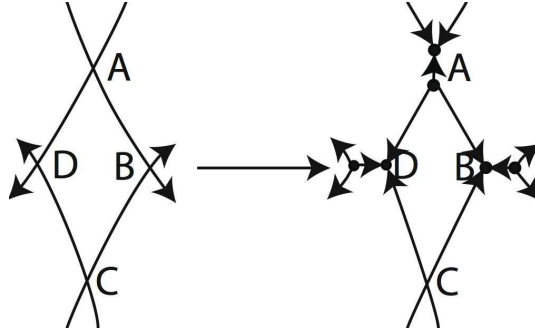


Figure 18: **Quadrilaterals of a special sort in K yield quadrilaterals in K_{us}**

Reidemeister move can be applied to K and no third Reidemeister move can be applied to K . Thus we arrive at the following natural conjecture about free knots.

Conjecture 1. *If a framed four-valent graph K has girth greater than or equal to 4 then this diagram is the unique minimal diagram for the knot class of K .*

In fact, the “minimality” part partially follows from Corollary 5. Indeed, all diagrams of girth 5 and many diagrams of girth 4 satisfy the conjecture. For a diagram of girth 4 to satisfy the conjecture, one should forbid the quadrilaterals of the following sort (see Fig. 18) which lead to quadrilaterals in K_{us} .

For such a quadrilateral, there are two opposite vertices (say, A and C) such that all the four edges of the quadrilateral emanate from A and C (and come into the other two vertices, B and D). Concerning the uniqueness of such minimal diagram, a very little is known because of the “matching ambiguity” discussed above.

6 Braids and the Kuperberg Bracket

In this section we use our graphical reduction method to give a graph-polynomial valued trace function on the Virtual Hecke algebra. Our procedure is analogous to the graphical trace on the Temperley-Lieb algebra [24] that is implicit in the bracket model for the Jones polynomial. Here we apply the same ideas to our graphical analysis of the Kuperberg bracket for virtual knots. In order to accomplish this, we first recall the formalism of the Kuperberg bracket as described in [5] and translate this into braid form. View Figure 19. Here we show the expansion formulas for the Kuperberg bracket. The reader can regard the expansion of a crossing as the expansion of a generator σ_i of the Artin braid group. Then we see that this expansion has the form

$$\sigma_i = A^2 I - A^{-1} P_i$$

where P_i stands for the double- Y -graph depicted in Figure 20. In that Figure we have shown how P_i is formed for $i = 1, 2, 3$ in the image of the braid group on 4 strands. We have also illustrated how

$$P_i^2 = (A^3 + A^{-3}) P_i$$

and in Figure 21 we have illustrated how the quadrilateral expansion formula for the Kuperberg bracket leads to the graphical identities:

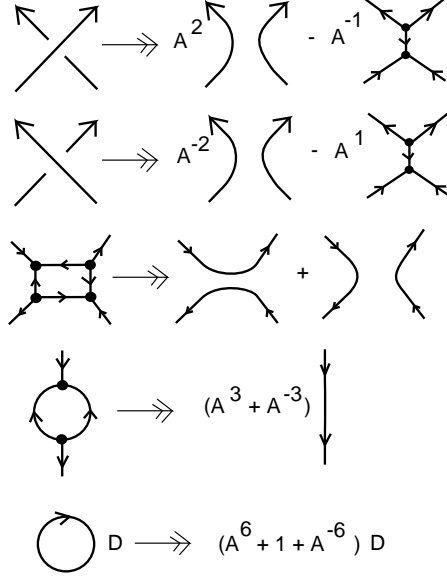


Figure 19: **Kuperberg Bracket**

1. $P_i P_{i+1} P_i = P_i + G_i$
2. $P_{i+1} P_i P_{i+1} = P_{i+1} + G'_i$
3. $G_i = G'_i$

Thus one can conclude the graphical identity

$$P_i P_{i+1} P_i - P_i = P_{i+1} P_i P_{i+1} - P_{i+1}.$$

and it is easy to see that

$$P_i P_j = P_j P_i$$

for $|i - j| > 2$.

We formalize this multiplication of graphs via the pattern shown in Figure 22. We make a category, *GraphCat*, by using graphs with free ends (nodes of order one) immersed in the plane. Other than the free ends, the graphs have trivalent vertices (oriented as in this paper with all edges pointing either in to a vertex or out from the vertex). They may have virtual crossings in their planar immersions. We include the circle among these graphs, but it is reduced to a scalar just as earlier in this paper. The ends of the graphs are arranged so that each graph is in a rectangle with its sides parallel to the standard vertical and horizontal directions. The free ends of the graph are either at the top of the box or at the bottom and they are oriented compatibly with the graph. A box with no free ends from the top or from the bottom is given a dotted line at the top or bottom just for notational purposes. The objects in *GraphCat* consist in the set $Obj = \{[0], [1], [2], \dots\}$ in 1 - 1 correspondence with the non-negative integers. A graph G with m free lower ends and n free upper ends is a morphism

$$G : [m] \longrightarrow [n].$$

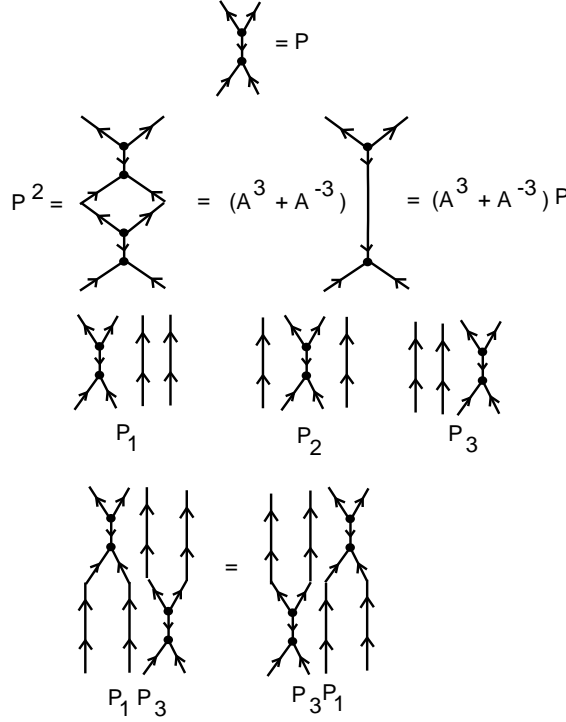


Figure 20: **Hecke Algebra 1**

The most general morphisms in *GraphCat* are linear combinations of graph-morphisms from $[n]$ to $[m]$ (n and m are fixed for a given linear combination) with coefficients in the ring $\mathbb{Z}[A, A^{-1}]$. Products are defined by taking the set of graph-morphisms as generators for a module over the ring $\mathbb{Z}[A, A^{-1}]$ and formally multiplying elements of this ring. Individual graph morphisms multiply by their categorical composition as described above.

The graph in the box is taken up to equivalences using the Kuperberg bracket expansion. This means that bigons and quadrilaterals are expanded out and any circles that result are evaluated. A morphism in this category is the equivalence class of such a box. Note that this equivalence class will often be a linear combination of graph classes. Morphisms are composed by direct multiplication (attachment of bottom ends to top ends as in the Figure) of graph-boxes, and the distributive law applies to the linear combinations. Note that two graph-boxes can be multiplied only if they have a matching number of free edges. We call this category *GraphCat*.

Recall that \mathcal{M} is the module $\mathbb{Z}[A, A^{-1}][t_1, t_2, \dots]$ of Laurent polynomials over formal commutative products of graphs from \mathcal{T} where \mathcal{T} denotes the set of equivalence classes of graphs without free ends. We define

$$\text{Trace} : \text{GraphCat} \longrightarrow \mathcal{M}$$

by the formula

$$\text{Trace}(G) = [[\bar{G}]]$$

where \bar{G} denotes the closure of G directly analogous to the standard closure of a braid, as shown in Figure 22 and $[[\bar{G}]]$ denotes the reduced equivalence class in \mathcal{M} of the linear graph combination

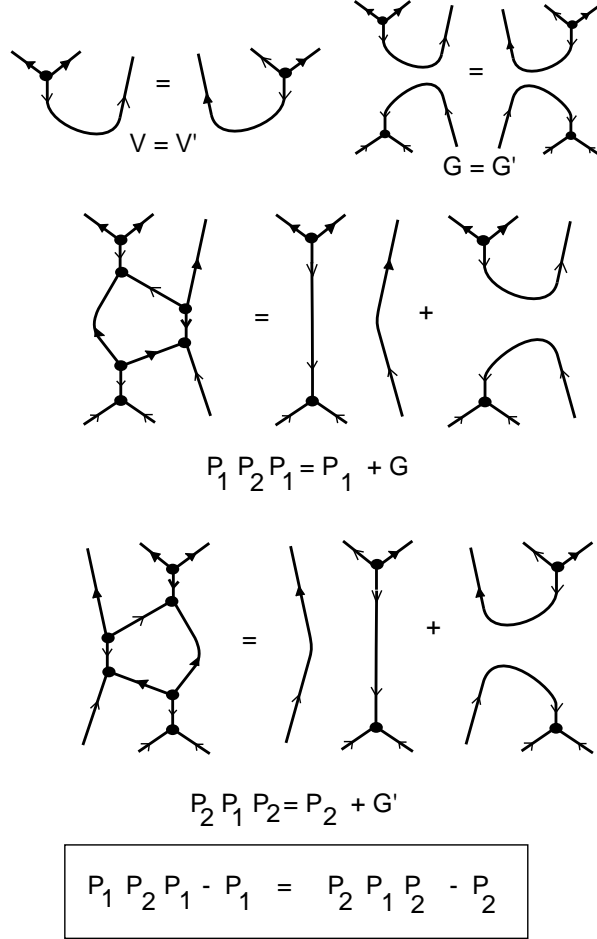


Figure 21: **Hecke Algebra 2**

\bar{G} .

From the point of view of the theory of braids the Hecke algebra $H_n(q)$ is a quotient of the group ring $\mathbb{Z}[q, q^{-1}][B_n]$ of the Artin braid group by the ideal generated by the quadratic expressions

$$\sigma_i^2 - z\sigma_i - 1 \quad (6)$$

for $i = 1, 2, \dots, n-1$, where $z = q - q^{-1}$. This corresponds to the identity $\sigma_i - \sigma_i^{-1} = z1$, which is sometimes regarded diagrammatically as a skein identity for calculating knot polynomials. By the same token, we define the *virtual Hecke algebra* $VH_n(q)$ to be the quotient of the group ring of the virtual braid group VB_n (See [16]) $\mathbb{Z}[q, q^{-1}][VB_n]$ by the ideal generated by Eqs. (6). Here the reader can verify that with $q = A^2$ and the algebraic expressions (I is the identity element in the algebra),

1. $\sigma_i = A^2 I - A^{-1} P_i$,
2. $\sigma_i^{-1} = A^{-2} I - A^1 P_i$,

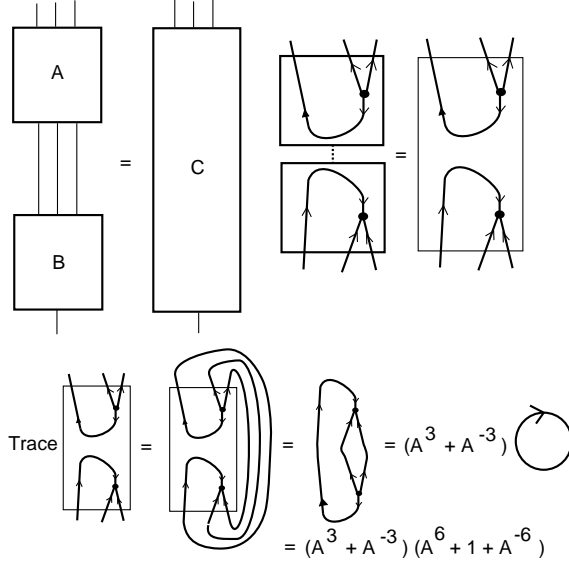


Figure 22: **Graph Category and Trace**

3. $P_i^2 = (A^3 + A^{-3})P_i$,
4. $P_i P_{i+1} P_i - P_i = P_{i+1} P_i P_{i+1} - P_{i+1}$.
5. $P_i P_j = P_j P_i$ for $|i - j| > 2$.

Using these definitions we arrive at the identical Hecke algebra that is obtained from the equation (6). The equations involving the elements P_i ensure that the braiding relations are satisfied by the images of the σ_i in the Hecke algebra. Of course there are virtual generators v_i in the virtual braid group and these satisfy relations that we shall omit to write here. The reader can consult [16] for this information. The virtual Hecke algebra also has virtual elements, that we call v_i . The virtual element v_i in VB_n consists in $i - 1$ parallel strands, then a virtual crossing of strands i and $i + 1$ and then parallel strands from strand $i + 2$ to strand n . We will assume that $v_i P_i = P_i v_i = P_i$ for this representation. This is in accord with our previous assumptions about the behaviour of virtual crossings that are adjacent to a trivalent graphical vertex. We let $VH_n = VH_n(q)$ denote the virtual Hecke algebra on n strands, written in the above form using P_i . In this form, we have a surjection

$$h : VB_n \longrightarrow VH_n$$

described by these equations. Note that we use σ_i both for the braid group generator and for its image in the Hecke algebra. In the Hecke algebra σ_i expands in terms of the identity and the projector P_i .

Note that in all diagrams involving the virtual Hecke algebra or involving virtual braids, when we speak of VB_n or VH_n , the number of free ends at the bottom is equal to the number of free ends at the top and is equal to n in both cases.

Along with this mapping of the virtual braid group to the virtual Hecke algebra we have a functor from the category of virtual braids to the category of graphs *GraphCat*,

$$g : B_n \longrightarrow \text{GraphCat},$$

given by expanding each braid generator according to the Kuperberg bracket as in Figure 19. It is clear at once from the discussion above that this map g factors through the virtual Hecke algebra so that we have

$$k : VH_n \longrightarrow \text{GraphCat}$$

taking the virtual Hecke algebra as a category on one object to a subalgebra of *GraphCat* and so that $g = k \circ h$. This means that for a virtual braid b with standard closure \bar{b} , the extended Kuperberg Bracket with values in M can be expressed by the formula below where k is the map defined above with domain the virtual Hecke algebra.

$$[[\bar{b}]] = \text{Trace}(g(b)) = \text{Trace}(k(h(b)))$$

Here $\text{Trace} : \text{GraphCat} \longrightarrow \mathcal{M}$ is the map defined at the beginning of this section.

The extended Kuperberg bracket gives a topologically invariant function on the virtual braid group, and it gives a trace function on the virtual Hecke algebra. This is the first time that such a trace function has been constructed for the virtual Hecke algebra. For the virtual Hecke algebra we define $Tr : VH_n \longrightarrow M$ by the formula

$$Tr(x) = \text{Trace}(k(x)).$$

In other words, we interpret basic products in the virtual Hecke algebra as graphs in *GraphCat* and we evaluate them by closure and the reduction via Kuperberg bracket relations. Note that $Tr(XY) = Tr(YX)$ for $X, Y \in VH_n$ because both XY and YX have the same closure. Other properties of this trace will be analyzed in a separate publication. We have shown here that our generalized Kuperberg bracket can be expressed in terms of this trace on the virtual Hecke algebra.

7 On the Penrose Coloring Bracket

A *proper edge 3-coloring* of a trivalent graph is an assignment of three colors (say from the set $\{0, 1, 2\}$) to the edges of the graph such that three distinct colors appear at every vertex. In [31] Penrose gives a bracket that counts the proper edge 3-colorings of a trivalent plane graph.

The Penrose bracket is defined as shown in Figure 23. The Penrose bracket is an evaluation of trivalent graphs immersed in the plane so that cyclic orders at the nodes are given. It can be defined independently of the planar immersion so long as the cyclic orders at the nodes are specified, but counts colorings correctly only for planar *embeddings*. Thus the Penrose bracket is well-defined for virtual graphs as we have defined them earlier in the paper. Note that in the expansion formula for the Penrose bracket we have one term with crossed lines and a virtual crossing. In completely expanding a graph by this formula one has many curves with virtual self-crossings. Each curve is evaluated as the number 3 and thus the evaluation of the Penrose bracket is a signed sum of powers of 3.

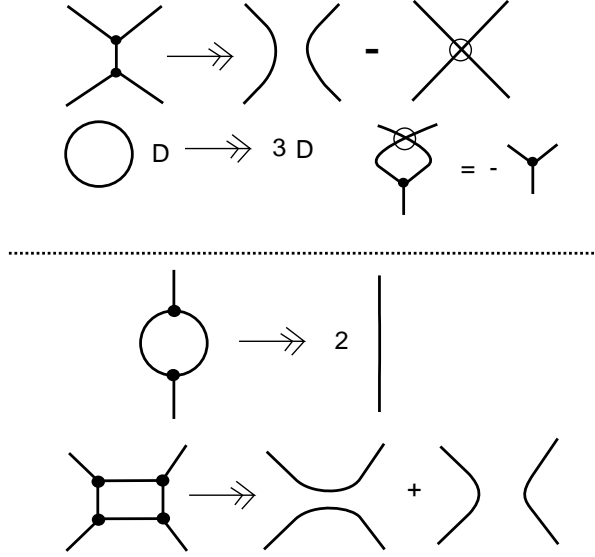


Figure 23: **Penrose Bracket and Identities**

Note in Figure 23 the identities that we have listed. These follow from the the expansion formulas above the line. It is then apparent that the evaluation of the Kuperberg bracket at $A = 1$ on an oriented trivalent plane graph is equal to the value of the Penrose bracket for this graph. The Kuperberg bracket is calculated by the same reduction formulas as the Penrose bracket, and for the oriented graph, we only need the bigon and quadrilateral reductions. This observation is the main reason we mention the Penrose bracket in this context.

In Figure 24 we show the other reduction identities for the Penrose bracket (for triangles, pendant loops and five-sided regions). These identities can be used to evaluate the Penrose bracket for unoriented plane graphs, since an Euler characteristic argument tells us that any trivalent graph in the plane with no isthmus must have a region with less than six sides. We can take

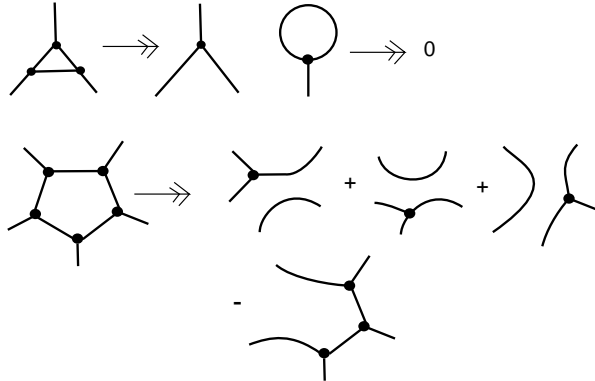


Figure 24: **Further Penrose Bracket Identities**

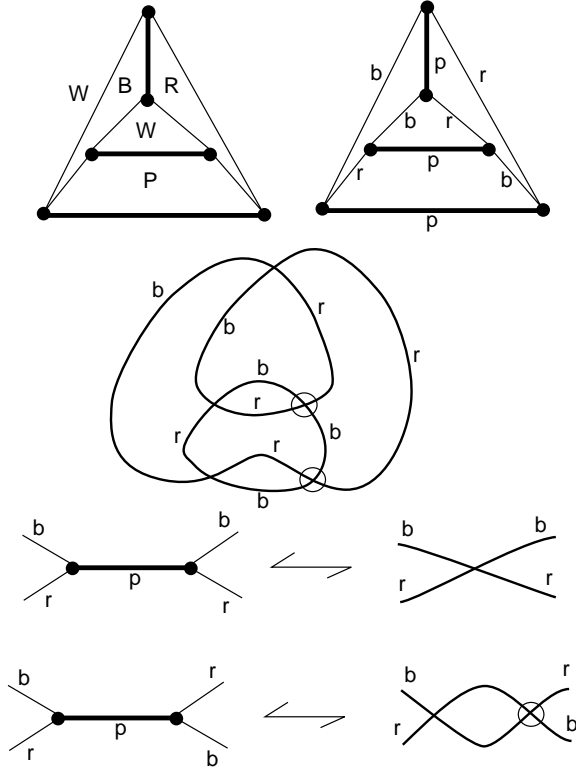


Figure 25: Coloring Graphs and Free Knots

the reduction formalism of the Penrose bracket and attempt to use it to create new invariants of free knots. This will be the subject of a separate paper. We are working on a similar project in relation to the Kuperberg G_2 bracket [3].

Finally, we point out a relationship between coloring maps, graphs and free knots and links. Examine Figure 25. At the top left of the figure we show a trivalent graph in the plane with a four-coloring of its regions by the colors $\{W, R, B, P\}$. Two regions that share an edge are colored with different colors. This graph is a simplest example of a plane graph that requires four colors. Adjacent to the coloring of the regions of this graph we have illustrated a coloring of its edges by the three colors $\{r, b, p\}$ so that every node of the graph has three distinct colored edges incident to it. The edge coloring can be obtained from the face coloring by regarding $\{W, R, B, P\}$ as isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ with W the identity element and $R^2 = B^2 = P^2 = W$ and $RB = BR = P, RP = PR = B, BP = PB = R$. Then the edge coloring is obtained by coloring each edge with the product of the region colors on either side of it.

Definition. Call a component in a free link diagram (represented as a planar diagram) *even* if there are an even number of virtual crossings on this component that occur between the given component and other components of the link. (See Section 2.3 for definitions related to framed and free knots and links.) Call a free link diagram *componentwise even* if every component of the link is even. Note that a single component free knot diagram is componentwise even since there are zero virtual crossings between the one component and any other component.

Below the two colored graphs in Figure 25 we illustrate a free link diagram that is obtained from the edge-colored graph above it by using the translations illustrated below the free link diagram. In these translations we use two-colors for the edges of a free link. *Opposite edges are colored by different colors from the binary set $\{r, b\}$.* We call a free link diagram 2-colored if it is colored in $\{r, b\}$ so that opposite edges have different colors. It is clear from the definition that *a free link diagram can be 2-colored if and only if it is componentwise even.* By using the translations in the figure we see that *an edge-colored trivalent graph immersed in the plane corresponds uniquely to a colored free link diagram.* In the figure we see that a colored planar graph may correspond to a free link diagram that has virtual crossings. The relationship between planar colorings and properties of free links needs further study.

References

- [1] M. Khovanov, L. Rozansky, (2008), Matrix Factorizations and Link Homology, *Fundamenta Mathematicae*, vol.199, pp. 1-91.
- [2] M. Khovanov, L. Rozansky, (2008), Matrix Factorizations and Link Homology II, *Geometry and Topology*, vol.12 (2008), pp. 1387-1425.
- [3] G. Kuperberg, (1994), The quantum G_2 link invariant, *Internat. J. Math.* **5**, pp. 61-85.
- [4] H. Murakami, T. Ohtsuki, S. Yamada, (1988), Homflypt polynomial via an invariant of colored planar graphs, *l'Enseignement mathématique* **44**, pp. 325-360.
- [5] T. Ohtsuki, "Quantum Invariants" , Vol. 29 - WS Series on Knots and Everything, World Scientific Pub. (2002).
- [6] V. O. Manturov, (2010), Parity in Knot Theory, *Sbornik Mathematics*, N.201, **5**, pp. 65-110.
- [7] M. Goussarov, M. Polyak and O. Viro, Finite type invariants of classical and virtual knots, *Topology* **39**, pp. 1045-1068. math.GT/9810073.
- [8] J. S. Carter, S. Kamada and M. Saito, Stable equivalences of knots on surfaces and virtual knot cobordisms. Knots 2000 Korea, Vol. 1, (Yongpyong), *JKTR* **11** (2002) No. 3, pp. 311-322.
- [9] H. A. Dye and L. H. Kauffman, Virtual crossing number and the arrow polynomial, *JKTR*, Vol. 18, No. 10 (October 2009), pp. 1335-1357. arXiv:0810.3858.
- [10] H. A. Dye, L.H. Kauffman, V. O. Manturov, (Chapter 4) On two categorifications of the arrow polynomial for virtual knots. In M. Banagl and C. Vogel, editors, *The Mathematics of Knots, Theory and Application*, Vol. 1, *Contributions in Mathematical and Computational Sciences*, pp. 95-124, Heidelberg University, Springer-Verlag, 2011. arXiv:0906.3408.
- [11] R. A. Fenn and V. G. Turaev. Weyl Algebras and Knots, *Journal of Geometry and Physics*.57 (2007) pp. 1313-1324, arXiv:math/0610481.
- [12] A. Bartholomew, R. A. Fenn, N. Kamada, S. Kamada, New invariants of long virtual knots, arXiv:0705.4639.
- [13] T. Kadokami, Detecting non-triviality of virtual links. *JKTR*, Vol. 12, No. 6 (2003), pp. 781-803.
- [14] N. Kamada and S. Kamada, Abstract link diagrams and virtual knots. *JKTR* **9** (2000), No. 1, pp. 93-106.

- [15] L. H. Kauffman, Knot diagrammatics. "Handbook of Knot Theory", edited by Menasco and Thistlethwaite, pp. 233–318, Elsevier B. V., Amsterdam, 2005. math.GN/0410329.
- [16] L. H. Kauffman and S. Lambropoulou, A categorical structure for the virtual braid group, *Communications in Algebra*, 39:12, (2011) pp. 4679-4704.
- [17] L. H. Kauffman, Virtual Knot Theory , *European J. Comb.* **20** (1999), pp. 663-690.
- [18] L. H. Kauffman, A Survey of Virtual Knot Theory in *Proceedings of Knots in Hellas '98*, World Sci. Pub. (2000) , pp. 143-202.
- [19] L. H. Kauffman, Detecting Virtual Knots, in *Atti. Sem. Mat. Fis. Univ. Modena Supple-mento al Vol. II*, pp. 241-282 (2001).
- [20] L. H. Kauffman, R. A. Fenn and V. O. Manturov, Virtual Knot Theory – Unsolved Problems, *Fund. Math.* **188** (2005), pp. 293–323, math.GT/0405428.
- [21] L. H. Kauffman, A self-linking invariant of virtual knots. *Fund. Math.* **184** (2004), pp. 135–158, math.GT/0405049.
- [22] L. H. Kauffman, An Extended Bracket Polynomial for Virtual Knots and Links. JKTR, Vol. 18, No. 10 (October 2009). pp. 1369 - 1422, arXiv:0712.2546.
- [23] L. H. Kauffman and V. O. Manturov, Parity in Virtual Knot Theory, (in preparation).
- [24] L.H. Kauffman, Knots and Physics, World Scientific Publishers (1991), Second Edition (1993), Third Edition (2002). Fourth Edition (2012).
- [25] G. Kuperberg, What is a virtual link? *Algebr. Geom. Topol.* **3** (2003), pp. 587 - 591 (elec-tronic). arXiv:math.GT/0208039 v1 5 Aug 2002.
- [26] V.O. Manturov, Khovanov homology of virtual knots with arbitrary coefficients, *Izvestiya: Mathematics* **71**:5, 967–999 (2007) (Original Russian Text in *Izvestiya RAN: Ser. Mat.* **71**:5, 111–148 (2007)). J. Knot Theory Ramifications 16 (2007), no. 3, pp. 345377.
- [27] V. O. Manturov, Parity in knot theory', *Math. sb.* **201**:5, 693–733 (2010) (Original Russian Text in *Mathematical sbornik* **201**:5, pp. 65–110 (2010)).
- [28] V. O. Manturov and D. Ilyutko, "Virtual Knots: The State of the Art", World Scientific Publishing Co., Series on Knots and Everything (2012).
- [29] D.P Ilyutko, V.O. Manturov, I.M. Nikonov, Virtual Knot Invariants Arising From Parities, arXiv.MATH/GT:1102.5081
- [30] V. O. Manturov, On virtual crossing numbers for virtual knots (to appear)
- [31] R. Penrose, Applications of negative dimensional tensors, in "Combinatorial Mathematics and Its Applications", edited by D. J. A. Welsh, Academic Press (1971), pp. 221 - 244.
- [32] V. G. Turaev, Virtual strings. *Ann. Inst. Fourier (Grenoble)* **54** (2004), no. 7, pp. 2455–2525 (2005).